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# Final Project Report
# on
# Applications of Multiparameter Bifurcations of Period Functions
# AF-AFOSR-89-0078

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July 24, 1991

## ABSTRACT

The objective of the research project was a mathematical analysis of multiparameter bifurcation problems which arise in the study of ordinary differential equations, especially, the bifurcation of critical points of period functions and the bifurcation of continuous families of periodic trajectories.

The focus of the first year of research was on bifurcations of critical points of period functions and on bifurcation of limit cycles. The main results on the bifurcation of critical points of the period function are the following: a complete analysis of the number of local critical points of the period function which can bifurcate near the origin in the space of quadratic systems which have a center at the origin, an analysis of the number of such critical points of the period function which bifurcate in the class of second order scalar differential equations of higher degree, and applications of these results to the analysis of two point nonlinear boundary value problems. For the limit cycle bifurcations, quadratic perturbations of linear and quadratic systems containing a linearizable center were analyzed. The main results are the following: In the linear case at most three unperturbed periodic trajectories give birth to a family of limit cycles under perturbation. Of the four possible quadratic families, one has the property that at most one unperturbed periodic trajectory gives birth to a family of limit cycles, while in the remaining families there are at most two such periodic trajectories.

The focus of the second year of research was on the bifurcation of continuous families of limit cycles in multiparameter time dependent systems. The main results of this research are an analysis of the number and positions of the subharmonic solutions of periodically excited oscillations near resonance when the unperturbed system has a limit cycle. These results give a mathematical foundation to the phenomenon of frequency entrainment or phase locking in a wide class of model equations. In particular, applications are made to determine the subharmonic response of a class of van der Pol oscillators.

During the time extension of the project, research on bifurcation of subharmonic solutions was continued. The main results of this research are a strong version of a subharmonic bifurcation theorem and several new results on bifurcation from degenerate families. In addition, a new result on the finiteness of the number of critical points of the period function for an analytic system was obtained and an application was made to the finiteness problem for two point nonlinear boundary value problems.

# 1 Introduction

The main objective of the research project is a mathematical understanding of the multiparameter bifurcations that occur in nonlinear systems of ordinary differential equations, especially, in those classes of ordinary differential equations used as models of dynamic physical processes. During the first year of the research program the investigation concentrated on the acquisition of basic knowledge about two types of bifurcations that are present in the analysis of multiparameter systems when a complete classification of the dynamics for all values of the parameters is required. These are the bifurcation of period functions and the bifurcation of limit cycles from unperturbed planar systems with a center. The motivation for studying these situations and the results obtained were detailed in the First Annual Project Report. During the second year of the research program we built upon the results and the insights obtained during the first year and extended the research to the consideration of forced oscillation problems. This line of inquiry has remained the main topic for research for the remainder of the project. Again the results obtained during the second year are explained in the Second Annual Project Report. The body of the present report consists of three sections. The first two sections are edited versions of the two annual reports while the third section explains the results obtained since the second annual report. Current references to research articles resulting from this project are given in §5 while invited presentations of these results are referenced in §6.

# 2 First Year of Research

Although realistic models of physical phenomena usually require many state variables which are related by partial differential equations, detailed mathematical analysis can only be accomplished when various simplifications are made. In particular, truncations of series representations, special solutions, and reduction to invariant manifolds often lead to ordinary differential equations which are low degree polynomials in a few of the state variables. Thus, two dimensional systems of polynomial differential equations are a natural environment for detailed mathematical analysis where a basic understanding of the dynamics is important for later application to the original model equations. Since the full range of nonlinear phenomena seems to be present when the nonlinear terms are of order two, it is very natural to consider quadratic plane systems as a test case. While this project is not confined to this case, most of the work is presented in this context.

The research on period functions is concerned with the following problem. Consider an analytic plane system of differential equations $\mathcal{E}_\lambda$,

$$\dot{x} = f(x, y, \lambda), \; \dot{y} = g(x, y, \lambda), \quad \lambda \in \mathbf{R}^N$$

which for each parameter value $\lambda$ contains a continuous one parameter family of periodic trajectories whose inner boundary is the origin of the phase plane. Further, assume there is a line segment (a Poincaré section) $\Sigma$ which is transverse to all of these periodic trajectories of $\mathcal{E}_\lambda$ for $\lambda$ near some $\lambda_*$. For $\xi$, the coordinate along $\Sigma$, the period function $(\xi, \lambda) \mapsto P(\xi, \lambda)$ assigns to the periodic trajectory of $\mathcal{E}_\lambda$ through $\xi$ its minimum period. The problem is to describe the critical points of $P$ as the parameter $\lambda$ varies. This problem is connected with several applications. One of the most important of these is the application to two point boundary value problems of Neumann type; the natural two point boundary value problem which arises in the study of the buckling of flexible rods. Our results on this problem are not yet complete. At present, there are no general methods which will determine the number of critical points of the period function globally on its domain. However, we have been able to give a complete analysis of the bifurcations which can occur locally, i.e., near a stationary point which forms the inner boundary of the family of periodic trajectories. We have applied this theory to all of the centers which occur in quadratic systems as well as the centers in differential systems which arise in second order "kinetic+potential" systems, i.e. systems of the form

$$\ddot{x} + x + \lambda_2 x^2 + \lambda_3 x^3 + \cdots + \lambda_N x^N = 0,$$

when either there are no even order terms in the equation or when $N \leq 6$. In each case we can obtain tight bounds on the number of critical points of the period function which can arise near the the center as $\lambda$ is varied. For example, in the quadratic case all bifurcations of the period function lead to bifurcations of at most two local critical points of the period function and in the case of the "kinetic+potential" systems with only odd order terms, if $N = 2n - 1$ then there are at most $n - 1$ local critical points of the period function which bifurcate. This work is published in the paper *Bifurcation of critical periods for plane vector fields.*

Perhaps the most important byproduct of our analysis of the critical points of period functions is our exposition of a method for obtaining bifurcation results for the number of zeros of an arbitrary analytic function $(\xi, \lambda) \mapsto F(\xi, \lambda)$ near $\xi = 0$, $\lambda = \lambda_*$ when $F$ and its partial derivatives with respect to $\xi$ of all orders vanish at $(0, \lambda_*)$. In the case of the period function, this arises when $F(\xi, \lambda) = P_\xi(\xi, \lambda)$ and the system $\mathcal{E}_\lambda$ at $\lambda = \lambda_*$ has an isochronous center at the origin. Here, the Implicit Function Theorem and the Weierstrass Preparation Theorem do not apply to $F$ near $(0, \lambda_*)$. Pursuant to the methods we have developed to overcome this difficulty, one is invariably lead to very lengthy calculations, especially, computations related to polynomial ideals. These computations are very efficiently carried out with the use of a computer algebra system. Since there is much current interest in the use of computer algebra in applied analysis, we decided to promote the general method for finding the zeros of an analytic multiparameter function near an infinite order zero and the attendant computer algebra in a separate publication. The results will appear in the proceedings of the important recent University of Cincinnati conference on Computer Assisted Proofs in Analysis, in our paper *On A Computer Algebra Aided Proof in Bifurcation Theory.*

For the bifurcation of limit cycles we again consider an analytic differential equation system depending on a parameter. But now, only the unperturbed system (given at $\lambda = 0$) is assumed to have a center surrounded by a one parameter family of periodic trajectories. The problem is to determine the number and position of those periodic trajectories surrounding the center of the unperturbed system which are the limiting members of continuous families of limit cycles which exist for the systems corresponding to small values of $\lambda \neq 0$. This problem is classical. It arises in the applications whenever the differential equation model is a perturbation of a conservative system. In order to ascertain the response of the perturbed system, it is necessary to analyze perturbation from centers, continuous families of periodic trajectories, separatrix cycles , etc. Here, many of the main mathematical foundations for the subject are well known, it is the application of these results to specific systems which is of current interest. Again, these applications often require extensive algebraic computations which are amenable to computer algebra. Our experience with computer algebra systems gained in the first part of the project proved very useful in our work on the oscillation problem. The main results obtained to date are for the quadratic systems, although the techniques employed will certainly be applicable to other situations. We have completely cataloged the number and position of the periodic trajectories at which a family of limit cycles emerges from a linearizable center of any quadratic system. For the quadratic systems, the linearizable center can be linear, or one of four nonlinear families. We consider arbitrary quadratic perturbations in each of the five cases and we show that at most three limit cycle families emerge from the periodic trajectories surrounding the linear center, at most one family emerges for one of the nonlinear cases and at most two arise in each of the remaining cases. This bifurcation analysis is carried out to all orders. Thus, the response of the system can be obtained, even when degeneracies are present, for any smooth path in the parameter space. This should be contrasted with the usual results which only consider first order bifurcation in the direction of a line in the parameter space. The notable extension provided by our work should prove valuable in the analysis of problems where degeneracies can not be avoided or where the parameters appear nonlinearly in the model equations. This research is published in the paper *Bifurcation of Limit Cycles from Quadratic Isochrones.*

# 3 Second Year of Research

One aspect of nonlinear dynamics of prime importance is the determination of the response of a nonlinear system to an excitation. This problem can take many forms as the excitation may be an external force or a dynamic change in some of the parameters of the system (a parametric excitation). The stimulus can also be of several types. For example, the stimulus can be impulsive, random or periodic. Each such situation leads to difficult and important mathematical questions. While we have not sought to attack all of these questions, our research is motivated by a desire to understand some aspects of the applications of nonlinear dynamics from this perspective. In particular, we have considered some important problems of this type that arise naturally when a periodic excitation is applied to a dynamical system whose free oscillation is a limit cycle.

For our analysis, we assume the unperturbed dynamical system is modeled by an autonomous multiparameter ordinary differential equation with a two dimensional state variable:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda), \quad \mathbf{x} \in \mathbf{R}^2, \quad \lambda \in \mathbf{R}^N.$$

This system is further assumed to support a self sustained oscillation, i.e., a periodic solution $\Gamma$ with period $T > 0$. Next, we consider the application of a periodic excitation

$$(\mathbf{x}, t, \lambda) \mapsto \mathbf{g}(\mathbf{x}, t, \lambda), \quad \mathbf{x} \in \mathbf{R}^2, \quad t \in \mathbf{R}, \quad \lambda \in \mathbf{R}^N$$

where $\mathbf{g}$ is periodic of period $\eta$, i.e., for all $(\mathbf{x}, t)$

$$\mathbf{g}(\mathbf{x}. t + \eta, \lambda) = \mathbf{g}(\mathbf{x}, t, \lambda)$$

and where the period is in resonance with the period of the unforced oscillation, i.e., there are positive integers $n$ and $m$ such that $nT = m\eta$. This leads to the model equation for the excited system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \lambda) + \epsilon \mathbf{g}(\mathbf{x}, t, \lambda), \quad x \in \mathbf{R}^2, \quad \lambda \in \mathbf{R}^N.$$

The response of the system to the resonant periodic excitation is the flow of this nonlinear system. Of particular interest are the periodic trajectories of this flow. Our research project seeks to determine the existence of some special periodic solutions, namely, the (sub)harmonic solutions of the differential equation for small values of the bifurcation parameter $\epsilon$, i.e., periodic solutions each with period an integer multiple of $\eta$, the period of the excitation. If asymptotically stable subharmonics exist, nearby orbits in their basins of attraction are said to be entrained to the subharmonic of the input frequency or one says they are phase locked to this frequency.

In order to determine the subharmonics, we take the geometric approach by considering an appropriate Poincaré map. For this, we view a point $\xi \in \mathbf{R}^2$ as an initial value for the differential equation and, for fixed $\lambda$, we let $t \mapsto \mathbf{x}(t, \xi, \epsilon)$ denote the solution of the $\epsilon$ dependent system starting at $\xi$. The parameterized Poincaré map $\mathbf{P}$ is the function given by $(\xi, \epsilon) \mapsto \mathbf{x}(m\eta, \xi, \epsilon)$. Now, if $\xi_0$ is on the periodic trajectory of the unperturbed system with period $T$, then $P(\xi_0, 0) = \xi_0$. Thus, it is natural to define the displacement function $\delta(\xi, \epsilon) := \mathbf{P}(\xi, \epsilon) - \xi$ so that $\delta(\xi_0, 0) = 0$ and the subharmonic solutions correspond to the zeros of $\delta$ for $\epsilon \neq 0$. To be more precise we say $\xi_0$ is a *subharmonic branch point* if $\xi_0$ lies on the periodic trajectory of the unperturbed system, there exists $\epsilon_0 > 0$ and a continuous curve $\epsilon \mapsto \sigma(\epsilon)$, defined for $|\epsilon| < \epsilon_0$, such that $\sigma(0) = \xi_0$ and $\delta(\sigma(\epsilon), \epsilon) \equiv 0$. This leads to our mathematically precise formulation of the problem of the determination of the response of the dynamical system to the excitation: Find the number and the position of the subharmonic branch points.

In order to find the subharmonic branch points we wish to use the Implicit Function Theorem, but it turns out that the Jacobian derivative of $\delta$ *with respect to* $\xi$ *is singular at* $(\xi_0, 0)$, so an immediate application of the Implicit Function Theorem is not possible. However, the difficulty can be surmounted by a Lyapunov-Schmidt reduction. This procedure reduces the problem of finding a subharmonic branch point to the computation of certain partial derivatives of the projection of the displacement $\delta$ onto the tangential and normal vector fields along the unperturbed periodic trajectory $\Gamma$. To obtain some appreciation of the results obtained we must mention two other functions. First, for the unperturbed system we consider an orthogonal trajectory through $\xi_0$ and define the transition time to be the function which assigns to each point in some neighborhood of $\xi_0$ the minimum time required for an unperturbed trajectory to return to this orthogonal trajectory. In case $\Gamma$ belongs to a one parameter family of periodic trajectories of the unperturbed system, the transition time function is the period function. (This is the main link between the first year of research on this project that was devoted to a study of certain aspects of the period function and the current research on the existence of subharmonics.) The second function is a generalization of the subharmonic Melnikov function that has been used previously to obtain the existence of subharmonic solutions in the special case when the unperturbed periodic solution is a member of a one parameter family of periodic solutions of the unperturbed system. The new function we require is defined by

$$\mathcal{C}(\xi) := \left[\frac{1-\beta}{\alpha}\mathcal{N} + \mathcal{M}\right](\xi)$$

where

$$\begin{aligned}
\beta(t) := \beta(t,\xi) &:= \exp\left(\int_0^t \operatorname{div} \mathbf{f}(\phi_s(\xi))\, ds\right), \\
\alpha(t) := \alpha(t,\xi) &:= \int_0^t \left\{\frac{1}{\|\mathbf{f}\|^2}\left[2\kappa\|\mathbf{f}\| - \operatorname{curl} \mathbf{f}\right]\right\}(\phi_\tau(\xi))\beta(\tau)\, d\tau, \\
\mathcal{N}(\xi) &:= \int_0^{m\eta}\left\{\frac{1}{\|\mathbf{f}\|^2}\langle \mathbf{f},\mathbf{g}\rangle - \frac{\alpha(s)}{\beta(s)}\mathbf{f}\wedge\mathbf{g}\right\}(\phi_s(\xi))\, ds,
\end{aligned}$$

$$\mathcal{M}(\xi) \;:=\; \int_0^{m\eta} \left\{ \frac{1}{\beta(s)} \mathbf{f} \wedge \mathbf{g} \right\} (\phi_s(\xi))\, ds$$

and where $\kappa$ is the curvature of the unperturbed periodic trajectory. We call $\mathcal{C}$ the *subharmonic bifurcation function.* In this notation it turns out that $\alpha$ is the derivative of the transition time map in the direction of the orthogonal trajectory, $\beta$ is the "characteristic multiplier" for the unperturbed periodic trajectory, and $\mathcal{M}$ is the subharmonic Melnikov function. We are now ready to state a version of our main result.

**Theorem 3.1 (Limit Cycle Subharmonic Bifurcation Theorem)** *Let $E_\epsilon$ denote the parameterized family of differential equations*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon \mathbf{g}(\mathbf{x}, t), \;\; \mathbf{x} \in \mathbf{R}^2, \;\; \epsilon \in \mathbf{R},$$

*such that $E_0$ has a limit cycle $\Gamma$ whose period is in resonance with the $\eta$-periodic external force $\mathbf{g}(\mathbf{x}, t)$, i. e. , there are relatively prime natural numbers $m$ and $n$ such that the period of $\Gamma$ is $m\eta/n$. If $\xi \in \Gamma$ is a simple zero of the subharmonic bifurcation function $\mathcal{C}$, i. e. , $\mathcal{C}(\xi) = 0$ and $d\mathcal{C}(\mathbf{f})(\xi) \neq 0$, and if $\alpha(\xi) \neq 0$, then $\xi$ is a subharmonic branch point.*

From the point of view of "pure" mathematics, the result is important because it gives a rigorous new method of justifying the existence of families of subharmonics. It is particularly satisfying to have a proof that rests firmly on the Implicit Function Theorem. From the point of view of new understanding, the result is very nice because it makes clear the role played by the derivative of the transition time, given by $\alpha(\xi)$, as a nondegeneracy condition for the bifurcation to subharmonic solutions and thus relates the subharmonic bifurcation problem to general results for the transition time functions of dynamical systems. From the point of view of applications, the result is useful because the bifurcation function $\mathcal{C}$ is given quite explicitly in terms of geometric quantities along the free oscillation of the unperturbed system. We also note that the theorem reduces to the classical case when the periodic trajectory of the unperturbed system belongs to a one parameter family of periodic trajectories. This is easy to see because in this case the "characteristic multiplier" $\beta$ is unity and therefore $\mathcal{C} = \mathcal{M}$, i.e., the bifurcation function $\mathcal{C}$ reduces to the subharmonic Melnikov function. The classical case will occur, for example, when the unperturbed system is Hamiltonian.

Our motivation for extending the usual Melnikov theory to the case when the unperturbed periodic trajectory is a limit cycle is our desire to obtain the subharmonic response for a wider class of systems than those which can be analyzed by the classic Melnikov theory. In particular, our theory is applicable to systems modeled by equations of van der Pol type. A typical example is provided by a system of the form

$$\begin{aligned}
\dot{u} &= v, \\
\dot{v} &= -u + a(1 - u^2)v \\
\dot{x} &= \tau y, \\
\dot{y} &= \tau(-x + a(1 - x^2)y) + \epsilon u
\end{aligned}$$

where $a$, $\tau$ and $\epsilon$ are real. Here, we view the $x$-$y$ system as our dynamical system with the excitation provided by the $u$-$v$ system via the coupling term $\epsilon u$ added to the $x$-$y$ system. For $\epsilon = 0$, $a > 0$ and $\tau > 0$, the $x$-$y$ system has a stable limit cycle as its free oscillation. If, in addition, $\tau$ is a rational number, then when $\epsilon \neq 0$ the $x - y$ system is excited by the resonant periodic output $t \mapsto u(t)$ of the $u - v$ system. The subharmonic response of the system, for small $\epsilon$, can be determined by applying the theorem. For a simple example, take $a = 2$ and $\tau = 1$ so there is a $1 : 1$ resonance. It can be shown using the theorem that there are two subharmonic branch points. One branch corresponds to a family of stable periodic solutions of the excited $x - y$ system parameterized by $\epsilon$ with each periodic solution of the family having the same period as the excitation (harmonic response entrained to the frequency of the unperturbed oscillation) while the other branch is a branch of unstable harmonics.

# 4 Research During Extension Period

During the final phase of the research the main focus has been a continuation of the work on subharmonic bifurcations completed during the second year of the project. In addition, a result on the finiteness of the number of critical points of the period function for an analytic vector field has been obtained. In order to describe the results, we will use some of the notation and terminology introduced in previous sections.

In §2 we described the Limit Cycle Subharmonic Bifurcation Theorem obtained during the second year of the project. Perhaps the most important result of the 'extension period' is a much stronger version of this theorem. For the statement of the theorem we require a modification of the definition of the bifurcation function which we previously denoted as $\mathcal{C}$. The new function we require is defined by

$$\mathcal{C}(\xi) := [(1-\beta)\mathcal{N} + \alpha\mathcal{M}](\xi).$$

We call $\mathcal{C}$ the *subharmonic bifurcation function.* The strengthened theorem applies either when the unperturbed period orbit $\Gamma$ is hyperbolic or when, at the bifurcation point, the derivative of the transit time does not vanish. In particular, if we know the periodic orbit is a hyperbolic limit cycle (the generic case) the nondegeneracy condition of the previous version of the theorem can be eliminated. Formally, we can state the new version of the theorem as follows:

**Theorem 4.1 (Limit Cycle Subharmonic Bifurcation Theorem)** *Let $E_\epsilon$ denote the parameterized family of differential equations*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{g}(\mathbf{x}, t) + \epsilon^2\mathbf{g}_R(\mathbf{x}, t, \epsilon), \;\; \mathbf{x} \in \mathbf{R}^2, \;\; \epsilon \in \mathbf{R},$$

*such that $E_0$ has a limit cycle $\Gamma$ whose period is in resonance with the $\eta$-periodic external force $\mathbf{G}(x, t, \epsilon) := g(x, t) + \epsilon\mathbf{g}_R(x, t, \epsilon)$, i. e. , there are relatively prime natural numbers $m$ and $n$ such that the period of $\Gamma$ is $m\eta/n$. If $\Gamma$ is hyperbolic and $\xi \in \Gamma$ is a simple zero of the subharmonic bifurcation function $\mathcal{C}$, i. e. , $\mathcal{C}(\xi) = 0$ and $d\mathcal{C}(\mathbf{f})(\xi) \neq 0$, then $\xi$ is a subharmonic branch point. Also, if $\xi \in \Gamma$ is a simple zero of the subharmonic bifurcation function and if $\alpha(\xi) \neq 0$, then $\xi$ is a subharmonic branch point.*

In view of the usual subharmonic bifurcation theory, where the unperturbed periodic orbit is contained in a one-parameter family on periodic trajectories and where the non vanishing of the derivative of the period function is the appropriate nondegeneracy condition for a bifurcation point to be a subharmonic branch point, we see that there is an interplay between the two types of degeneracy given by the number of vanishing derivatives of the transit time function and the multiplicity of the periodic trajectory. Actually, a theory can be developed to handle all the possible degeneracies. The simplest versions of the appropriate results in the next most degenerate cases are reasonably easy to state.

For these cases there is the possibility that more than one family of subharmonics is found near a subharmonic branch point. We will say $\xi \in \Gamma$ is a *subharmonic branch point with n-branches* if there is an $\epsilon_0 > 0$ and distinct (germs of) curves (at $\epsilon = 0$), $\epsilon \mapsto \sigma_k(\epsilon)$; $k = 1, \ldots, n$, each defined either for $\epsilon_0 < \epsilon \leq 0$, or for $0 \leq \epsilon < \epsilon_0$, and each with image in the section $\Sigma$, such that $\sigma_k(0) = \xi$ and $\delta(\sigma_k(\epsilon), \epsilon) \equiv 0$. The next theorem gives the result for the case of the period annulus.

**Theorem 4.2 (Order 2 Subharmonic Bifurcation Theorem)** *Let $E_\epsilon$ denote the parameterized family of differential equations*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{g}(\mathbf{x}, t) + \epsilon^2\mathbf{g}_R(\mathbf{x}, t, \epsilon), \ \mathbf{x} \in \mathbf{R}^2, \ \epsilon \in \mathbf{R},$$

*such that $E_0$ has a period annulus $\mathcal{A}$ and a periodic trajectory $\Gamma \subset \mathcal{A}$ that is in resonance with the $\eta$-periodic external force $\mathbf{G}$, i. e. , there are relatively prime natural numbers $m$ and $n$ such that the period of $\Gamma$ is $m\eta/n$. If $\Gamma$ is critical ($\alpha(\xi) \equiv 0$) and if $\xi \in \Gamma$ is a simple zero of the subharmonic Melnikov function, such that*

$$\mathcal{N}(\xi)d\alpha(\xi)\mathbf{f}^\perp(\xi) \neq 0,$$

*then $\xi$ is a subharmonic branch point with two branches. Moreover, these two branches exist only in the direction of $\epsilon$ such that*

$$\epsilon\mathcal{N}(\xi)d\alpha(\xi)\mathbf{f}^\perp(\xi) < 0.$$

In the limit cycle case we have the analogous theorem.

**Theorem 4.3 (Order 2 Limit Cycle Subharmonic Bifurcation Theorem)** *Let $E_\epsilon$ denote the parameterized family of differential equations*

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{g}(\mathbf{x}, t) + \epsilon^2\mathbf{g}_R(\mathbf{x}, t, \epsilon), \ \mathbf{x} \in \mathbf{R}^2, \ \epsilon \in \mathbf{R},$$

*such that $E_0$ has a periodic trajectory $\Gamma \subset \mathcal{A}$ that is in resonance with the $\eta$-periodic external force $\mathbf{G}$, i. e. , there are relatively prime natural numbers $m$ and $n$ such that the period of $\Gamma$ is $m\eta/n$. If $\xi \in \Gamma$ is such that following three conditions are satisfied: (i) $\alpha(\xi) = 0$ and $\beta(\xi) = 1$, (ii) either $\mathcal{M}(\xi)d\beta(\xi)\mathbf{f}^\perp(\xi) \neq 0$ or $\mathcal{N}(\xi)d\alpha(\xi)\mathbf{f}^\perp(\xi) \neq 0$, and (iii) $\xi \in \Gamma$ is a simple zero of the bifurcation function*

$$\mathcal{D} := \mathcal{N}(\xi)d\beta(\xi)\mathbf{f}^\perp(\xi) - \mathcal{M}(\xi)d\alpha(\xi)\mathbf{f}^\perp(\xi),$$

*then $\xi$ is a subharmonic branch point with two branches. Moreover, these two branches exist, in case $\mathcal{M}(\xi)d\beta(\xi)\mathbf{f}^{\perp}(\xi) \neq 0$, only in the direction of $\epsilon$ such that*

$$\epsilon\mathcal{M}(\xi)d\beta(\xi)\mathbf{f}^{\perp}(\xi) < 0$$

*and, in case $\mathcal{N}(\xi)d\alpha(\xi)\mathbf{f}^{\perp}(\xi) \neq 0$, only in the direction of $\epsilon$ such that*

$$\epsilon\mathcal{N}(\xi)d\alpha(\xi)\mathbf{f}^{\perp}(\xi) < 0.$$

As an application of the Order 2 Limit Cycle Subharmonic Bifurcation Theorem consider the oscillator $E_\epsilon$ given by

$$\dot{x} = y - x(1 - x^2 - y^2)^2 - \epsilon\cos t, \qquad \dot{y} = -x - y(1 - x^2 - y^2)^2 + \epsilon\sin t.$$

which has the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{g}(t).$$

Here, the unperturbed system has a (semi-stable) multiplicity 2 limit cycle $\Gamma$ on the unit circle. The corresponding integral curve of $E_0$ starting at $\xi := (\xi_1, \xi_2)$ is

$$x(t) = \xi_1\cos t + \xi_2\sin t, \quad y(t) = -\xi_1\sin t + \xi_2\cos t.$$

To apply the theorem, one can compute all of the geometric quantities explicitly. In particular, with respect to the coordinate $\xi := (\xi_1, \xi_2) \in \Gamma$,

$$\begin{array}{ll} \alpha(t,\xi) \equiv 0 & d\alpha(\xi)\mathbf{f}^{\perp}(\xi) = 0, \\ \beta(\xi) = 1, & d\beta(\xi)\mathbf{f}^{\perp}(\xi) = 16\pi. \end{array}$$

Also,

$$\begin{aligned} \mathcal{M}(\xi) &= \int_0^{2\pi} y\sin t - x\cos t\,dt = -2\pi\varsigma_1 \\ \mathcal{D}(\xi) &= -32\pi^2\xi_2. \end{aligned}$$

Thus, $\mathcal{D}$ has a simple zero along $\Gamma$ at $(\pm 1, 0)$ and, since the only zeros of $\mathcal{M}(\xi)d\beta(\xi)\mathbf{f}^{\perp}(\xi)$ are the points $(0, \pm 1)$, the theorem shows each of the bifurcation points is a subharmonic bifurcation point with two branches. At $(-1, 0)$ the harmonics exist for sufficiently small $\epsilon > 0$ while at $(1, 0)$ they exist for sufficiently small $\epsilon < 0$.

The final result we mention is the finiteness result for the number of critical points of the period function for an analytic vector field. This question arises from our study of critical points of the period function in relation to the bifurcation theory just described and in relation to the solutions of boundary value problems as described in §1. There are a number of different results which are formalized in the preprint *Finiteness for Critical Periods of Planar Analytic Vector Fields*, but perhaps the sharpest result is the following. We consider the finiteness problem for the important class of plane polynomial vector fields given as classical one degree of freedom Hamiltonian vector fields when the potential energy is a polynomial of degree $N$. In particular, assume the Hamiltonian has the form

$$H(x,y) = \frac{1}{2}y^2 + V(x)$$

where

$$V(x) = a_0 + a_1 x + \cdots + a_N x^N.$$

The Hamiltonian vector field is then given by the differential equations

$$\dot{x} = -\frac{\partial H}{\partial y} = -y, \qquad \dot{y} = \frac{\partial H}{\partial x} = V'(x).$$

**Theorem 4.4** *If a one degree of freedom Hamiltonian system with a polynomial potential has infinitely many critical periods, then the potential is quadratic and the Hamiltonian system is linear.*

The theorem and the ideas contained in its proof can be used to show the following: If a two point boundary value problems of either Dirichlet or Neumann type for the equation

$$\ddot{x} + V'(x) = 0,$$

where the potential $V$ is a polynomial, has infinitely many solutions with a specified number of nodes, then the equation must be linear.

# 5 Publications

The following publications are a direct result of the research project to date.

1. Carmen Chicone and Marc Jacobs, *Bifurcation of critical periods for plane vector fields*, Transactions of the American Mathematical Society, **312**(1989), 433–486.

2. Carmen Chicone and Marc Jacobs, *On a computer aided proof in bifurcation theory, Computer Aided Proofs in Analysis*, K. Meyer and D. Schmidt Eds., IMA Volumes in Mathematics and Its Applications, Vol 28, Springer-Verlag, New York, 1991, 52-70.

3. Carmen Chicone and Marc Jacobs, *Bifurcation of limit cycles from quadratic isochrones,* Journal of Differential Equations, **91**(1991), 268-327.

4. Carmen Chicone, *Bifurcations of nonlinear oscillations and frequency entrainment near resonance,* submitted for publication.

5. Carmen Chicone and Freddy Dumortier, *Finiteness for Critical Periods of Planar Analytic Vector Fields,* submitted for publication.

# 6 Interactions

Most of the results of this research project have been presented at professional meetings and invited addresses at universities. In fact, the following presentations have been given by PI C. Chicone as a direct result of the research project.

1. *Bifurcation of critical periods of plane vector fields,* University of Cincinnati, November 1988.

2. *The period function for plane vector fields,* Limburgs Universitair Centrum, Diepenbeek, Belgium, January 1989.

3. *The problem of the isochrone,* Limburgs Universitair Centrum, Diepenbeek, Belgium, January 1989.

4. *The period function for planar Hamiltonians,* Limburgs Universitair Centrum, Diepenbeek, Belgium, February 1989.

5. *Bifurcation of period functions,* Dynamical Systems Conference, Delft, The Netherlands, January 1989.

6. *Bifurcation of period functions,* Université de Bourgogne, Dijon, France, March 1989.

7. *Bifurcation of plane vector fields,* Computer Aided Proofs in Analysis, Cincinnati, Ohio, March 1989.

8. *Bifurcation of plane vector fields,* Workshop on Qualitative Theory of Vector Fields, Université de Montréal, August 1989.

9. *Bifurcation from isochrones,* CIRM Conference on Bifurcation and Periodic Orbits of Plane Vector Fields, Luminy, France, September 1989.

10. *Bifurcation of limit cycles for plane vector fields,* $19^{th}$ Midwest Differential Equations Conference, Rolla MO, October, 1990.

11. *Bifurcations of nonlinear oscillations*, Midwest Dynamical Systems Conference, Northwestern University, Evanston, March 1991.

12. *Bifurcation of subharmonics*, Northwestern University, Evanston, June 1991.